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On the structure of Hamiltonian operators in field theory

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Abstract. *A complete description of the Hamiltonian operators is shown and a «Darboux lemma» is proved (for some values of the parameters) in the framework of systems with infinite degrees of freedom.*

1. INTRODUCTION

The conception of the Hamiltonian formalism as it is well known now, has in its origin the notion of the Poisson bracket (see [1] for a motivation). In finite dimensional mechanics this means the following. Let M be the phase space of a mechanical system under consideration and $\mathscr{F} = C^\infty(M)$. The Poisson bracket structure on M is just a local Lie algebra structure on the real vector space $\mathscr{F}$. Denoting the corresponding Lie algebra operation by $\{f, g\} \in \mathscr{F}$ for $f, g \in \mathscr{F}$ we have

$$\{f, g\} = \{-g, f\} \qquad \text{(skew-symmetry)},$$

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \qquad \text{(the Jacobi identity)}.$$

«Local» means here that the operation $(f, g) \mapsto \{f, g\}$ is bidifferential, i.e. operators $X_f : \mathscr{F} \to \mathscr{F}$, $X_f(g) = \{f, g\}$, are differential for all $f \in \mathscr{F}$. In fact, it turns out that all operators X_f are of the first order [2]. Therefore a Poisson bracket on M may be introduced via the differential operator of the first order

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$\Gamma : \mathcal{F} \to D(M)$, where $D(M)$ denotes the $\mathcal{F}$-module of all C^∞ vector fields on M and $\Gamma(f) = X_f$. Thus,

$$\{f, g\} = \Gamma(f)(g). \tag{1}$$

By this reason an operator $\Gamma : \mathcal{F} \to D(M)$ is called Hamiltonian if the bracket $\{ \, , \, \}$defined by (1) is the Poisson bracket. Immediately the following problem arises: to classify (locally) Hamiltonian operators under diffeomorphisms.

The famous «Darbous lemma» in its Hamiltonian form asserts that two non-degenerate Hamiltonian operators are locally equivalent if their underlying manifolds have the same dimension. By a non-degenerate operator we understand one satisfying the condition: $\Gamma(f)_x = 0$ iff $d_x f = 0, f \in \mathcal{F}, x \in M$. It is worthy to point out that any non-degenerate Hamiltonian operator naturally determines a symplectic structure on its underlying manifold, and conversely. Regular degenerate Hamiltonian operators also may be described [1], [3].

In this paper we analyze the above problem for systems with infinite degrees of freedom or, in physical terms, for fields. Surely, it is much more difficult in this case. E.g., it is not trivial here to find the right formulation of «the Darboux lemma».

Our main results are the complete description of the Hamiltonian operators and the proof of «the Darboux lemma» for some small values of n = the number of independent variables, m = the number of dependent ones, and K = order of the operator. The greater part of these was announced without proof in [4], [5]. In what follows, all manifolds, fiberings, maps, etc., are supposed to be C^∞.

2. PRELIMINARIES

In this section we describe necessary notions and notations.

Let $\pi : E \to M$ be a fibering, $\dim M = n$, $\dim E = m + n$, and $\mathrm{Sec}(\pi)$ be the set of local sections of π. There are natural fiberings $\pi_k : J^k(\pi) \to M$, $\pi_{k,s} : J^k(\pi) \to J^s(\pi)$, $0 \leqslant s \leqslant k \leqslant \infty$, where $J^k(\pi)$ denotes the k-jet manifold of π. For $f \in \mathrm{Sec}(\pi)$ we denote its k-jet at a point $x \in M$ by $[f]_x^k$ and the corresponding section of π_k by $j_k(f)$. Obviously, $\pi_{k,s} \circ j_k(f) = j_s(f)$.

Let $x_1, \ldots, x_n, u^1, \ldots, u^m$ be local coordinates on E, x_i being a base coordinate and u^j being a fibre one. Then

$$x_1, \ldots, x_n, \ldots, u^i = p_0^i, \ldots, p_\sigma^i, \ldots, 1 \leqslant i \leqslant m, \quad |\sigma| \leqslant k,$$

are local coordinates on $J^k(\pi)$. Here $\sigma = (\sigma^1, \ldots, \sigma^n) \in \mathbb{N}^n$ is a multi-index, $|\sigma| = \sigma^1 + \ldots + \sigma^n$, and the functions p_σ^i are defined by equalities $p_\sigma^i \circ j_k(f) = \partial^\sigma f^i$, where $u^i = f^i(x)$ are the local equations of f. If $n = 1$, we write p^i instead of p_1^i and x instead of x_1. The same is about upper indices. Sometimes we also omit π if there is no risk of ambiguity.

«The manifold» $J^\infty(\pi)$ is the inverse limit of the sequence $\ldots \to J^k(\pi) \xrightarrow{\pi_{k,k-1}} \ldots \xrightarrow{\pi_{1,0}} J^0(\pi) = E$ and by the algebra $\mathscr{F}(\pi)$ of smooth functions on $J^\infty(\pi)$ we understand the direct limit of the algebra homomorphisms $\pi^*_{k,k-1} : C^\infty(J^{k-1}(\pi)) \to (C^\infty(J^k(\pi))$. Introducing subalgebras $\mathscr{F}_k(\pi) = \pi^*_{\infty,k}(C^\infty(J^k(\pi)) \subset \mathscr{F}(\pi)$, $k = 0, 1, \ldots$, we see that $\mathscr{F}_s(\pi) \subset \mathscr{F}_k(\pi)$, $s \leqslant k$, and therefore the algebra $\mathscr{F}(\pi)$ is filtered by its subalgebras $\mathscr{F}_k(\pi)$. Similarly, the $\mathscr{F}(\pi)$-module $\Lambda^i = \Lambda^i(J^\infty(\pi))$ of differential forms of degree i on $J^\infty(\pi)$ is defined as the direct limit of $C^\infty(J^k(\pi))$-modules $\Lambda^i(J^k(\pi))$ by maps $\pi^*_{k,k-1}$.

Let $A = \cup A_k$ be a filtered algebra, then $P = \cup P_k$ is a filtered A-module, if P_k is an A_k-module and $\ldots \subset P_k \subset P_{k+1} \subset \ldots$. For example, Λ^i is a filtered $\mathscr{F}(\pi)$-module. If A is commutative and $P = \cup P_k$, $Q = \cup Q_k$ are filtered A-modules, then a linear differential operator $\Delta : P \to Q$ over A, [6], is said to be filtered if for any k, $\Delta(P_k) \subset Q_{k+s}$ for some s depending on k. Below, we consider only filtered differential operators over $\mathscr{F}(\pi)$ and denote by $\mathrm{Diff}_k(P, Q)$ the set of filtered linear differential operators of order $\leqslant k$ acting from P to Q, P, Q being filtered $\mathscr{F}(\pi)$-podules. Obviously, $\mathrm{Diff}_k(P, Q) \subset \mathrm{Diff}_s(P, Q)$, $k \leqslant s$. Denote also $\mathrm{Diff}\,(P, Q) = \cup\, \mathrm{Diff}_k(P, Q)$.

In local coordinates on $J^\infty(\pi)$ described above a «scalar» operator $\Delta \in \mathrm{Diff}_k(\mathscr{F}(\pi), \mathscr{F}(\pi))$ may be presented as $\sum a^{i_1 \ldots i_r j_1 \ldots j_n}_{\sigma_1 \ldots \sigma_r}\, \partial^l / \partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_r}_{\sigma_r}\, \partial x_1^{j_1} \ldots \partial x_n^{j_n}$, summing by $1 \leqslant i_1, \ldots, i_r \leqslant m, j_1, \ldots, j_n \in \mathbb{Z}_+, \sigma_1, \ldots, \sigma_r \in \mathbb{N}^n$, $0 \leqslant r + j \leqslant k$, $j = \sum j_s$, with coefficients from $\mathscr{F}(\pi)$. If P and Q are free $\mathscr{F}(\pi)$-modules over the local chart, then any $\Delta \in \mathrm{Diff}\,(P, Q)$ may be presented by an operator matrix with its entries being scalar differential operators.

$D_k = \dfrac{\partial}{\partial x_k} + \sum\limits_{i,\sigma} p^i_{\sigma+\epsilon(k)} \dfrac{\partial}{\partial p^i_\sigma}$ is the total derivative in x_k, where $\epsilon(k) = (0, \ldots, 0, 1, 0, \ldots, 0)$ has 1 as its k-th component. Scalar operators locally expressible in the from $\sum\limits_\sigma a_\sigma D^\sigma$, where $a_\sigma \in \mathscr{F}(\pi)$, $D^\sigma = D_1^{\sigma_1} \circ \ldots \circ D_n^{\sigma_n}$, and matrix operators with such entries are called *C-differential*. The set of C-differential operators $P \to Q$ is denoted $C\,\mathrm{Diff}\,(P, Q)$. Intrinsicly, an operator is C-differential if it can be restricted on every submanifold of $J^\infty(\pi)$ having the form im $j_\infty(f)$, $f \in \mathrm{Sec}\,(\pi)$.

The $\mathscr{F}(\pi)$-module $D = D(J^\infty(\pi)) = \{\Delta \in \mathrm{Diff}_1(\mathscr{F}(\pi), \mathscr{F}(\pi)) : \Delta(1) = 0\}$ consists of all derivations of the algebra $\mathscr{F}(\pi)$ interpreted as vector fields on $J^\infty(\pi)$. Let $CD = D \cap C\,\mathrm{Diff}\,(\mathscr{F}(\pi), \mathscr{F}(\pi))$ and $D_C = D_C(\pi) = \{X \in D : [X, CD] \subset CD\}$.

Then D_C is a Lie algebra with the Lie operation being usual commutator, and CD is its ideal. So, we can define the Lie algebra $\varkappa(\pi) = D_C / CD$ interpreted as the algebra of vector fields on the «manifold» $\mathrm{Sec}\,(\pi)$, [6].

In a local chart any field $X \in CD$ has the form $\sum a_k D_k$, $a_k \in \mathscr{F}(\pi)$, and any field $X \in D_{\mathcal{C}}$ can be uniquely presented as $X = \text{Э}_f + Y$, $Y \in \mathcal{C}D$, where $f = (f_1, \ldots, f_m)$, $f_i \in \mathcal{F}(\pi)$, and $\text{Э}_f = \sum_{\sigma, i} D^\sigma(f_i)\, \partial/\partial p_\sigma^i$ is called an evolution with a generating function f. Therefore in the local chart the algebra $\varkappa(\pi)$ may be identified with the Lie algebra of all evolution derivations. We have $[\text{Э}_f, \text{Э}_g] = \text{Э}_{[f,g]}$, where $[f,g] = \text{Э}_f(g) - \text{Э}_g(f)$ is the higher Jacobi bracket and $\text{Э}_f(g) = (\text{Э}_f(g_1), \ldots, \text{Э}_f(g_m))$. In addition, the Lie algebra $\varkappa(\pi)$ is an $\mathcal{F}(\pi)$-module. Identifying (locally) elements of $\varkappa(\pi)$ with evolution derivations we have $\varphi\, \text{Э}_f = \text{Э}_{\varphi f}$, $\varphi \in \mathcal{F}(\pi)$.

Next we need to define «functions» on the «manifold» Sec (π) to get Hamiltonian formalism on it. To do that introduce $\mathcal{F}(\pi)$-modules

$$\mathcal{C}\Lambda^i = \{\omega \in \Lambda^i : j_\infty(f)^*(\omega) = 0, \forall f \in \operatorname{Sec}(\pi)\}.$$

Evidently, $\mathcal{C}\Lambda = \Sigma\, \mathcal{C}\Lambda^i$ is a d-closed ideal in $\Lambda = \Sigma \Lambda^i$. In particular, this allows us to define the factor-operator $\bar{d} : \bar{\Lambda}^i \to \bar{\Lambda}^{i+1}$ of $d : \Lambda^i \to \Lambda^{i+1}$, where $\bar{\Lambda}^i = \Lambda^i / \mathcal{C}\Lambda^i$. Locally elements of $\bar{\Lambda}^i$ may be identified with π_∞-horozintal forms on $J^\infty(\pi)$ and expressed as $\Sigma a_{k_1 \ldots k_i}\, \bar{d}x_{k_1} \wedge \ldots \wedge \bar{d}x_{k_i}$, $a_{k_1 \ldots k_i} \in \mathcal{F}(\pi)$. In these terms the operator $\bar{d}$ acts as $\bar{d}(f\, \bar{d}x_{k_1} \wedge \ldots \wedge \bar{d}x_{k_i}) = \bar{d}f \wedge \bar{d}x_{k_1} \wedge \ldots \wedge \bar{d}x_{k_i}$, $\bar{d}f = \sum_k D_k(f)\, \bar{d}x_k$. Now we define «the function space on Sec (π)» to be $L = \bar{\Lambda}^n / \bar{d}\bar{\Lambda}^{n-1}$. By above locally elements of $\bar{\Lambda}^n$ may be considered as Lagrangian densities, while their equivalence classes modulo $\bar{d}\bar{\Lambda}^{n-1}$ as «actions», i.e. functionals on Sec (π). This gives a motivation for L.

Finally, we have to understand how «vector fields» act on «functions» on Sec (π). Let $X(\omega)$ denote the Lie derivatives of $\omega \in \Lambda^i$ along $X \in D$ and $\bar{\omega} = \omega + \mathcal{C}\Lambda^i$ denote the element of $\bar{\Lambda}^i$ corresponding to ω. If $X \in \mathcal{C}D$ then «the infinitesimal Stokes' formula» $X(\omega) = X \rfloor d\omega + d(X \rfloor \omega)$ reduces to the formula $X(\bar{\omega}) = X \rfloor \bar{d}\bar{\omega} + \bar{d}(X \rfloor \bar{\omega})$, where $X(\bar{\omega}) = \overline{X(\omega)}$ and $X \rfloor \bar{\rho} = \overline{X \rfloor \rho}$. It shows that $X(\bar{\omega}) = \bar{d}(X \rfloor \bar{\omega})$ for $\bar{\omega} \in \bar{\Lambda}^n$, because $\bar{\Lambda}^{n+1} = 0$.

Therefore, the formula $\chi(\Omega) = [X(\bar{\omega})]$ defines correctly the *Lie derivative* of $\Omega = [\bar{\omega}] \in L$ along $\chi = X + \mathcal{C}D \in \varkappa(\pi)$, where $[\rho]$ denotes the element of L corresponding to $\rho \in \bar{\Lambda}^n$ by the natural projection $\bar{\Lambda}^n \to L$.

3. HAMILTONIAN OPERATORS

As in the classical finite-dimensional case a Hamiltonian operator on Sec (π) must act from L («functions») to $\varkappa$ («vector fields»). Of course, such an operator must be local, i.e. differential. But L is not an $\mathcal{F}(\pi)$-module. So, the usual notion of a differential operator acting on L is meaningless. However, $\bar{\Lambda}^n$ is an $\mathcal{F}(\pi)$-module. Therefore, a differential operator on L may be understood as a differential operator on $\bar{\Lambda}^n$ vanishing on $\bar{d}\bar{\Lambda}^{n-1}$. Now we notice that the Euler operator $\mathscr{E}$ i.e. the operator assigning to Lagrangian densities corresponding Euler-Lagrange equations, vanishes on $\bar{d}\Lambda^{n-1}$. Moreover, in a sense which will not be discussed here, it is the universal one in the class of operators vanishing on $\bar{d}\Lambda^{n-1}$. It can be shown that the range of E is $\hat{\varkappa}$, where $\hat{P} = \mathrm{Hom}_{\mathscr{F}}(P, \bar{\Lambda}^n)$ for an $\mathscr{F}$-module P. Therefore, differential operators on $\bar{\Lambda}^n$, which may be treated as operators on L, can be supposed having the form $\nabla \circ \mathrm{E}$, where ∇ is a differential operator on $\hat{\varkappa}$. Moreover, the operator ∇ must be $\mathcal{C}$-differential in order that $\nabla \circ \mathrm{E}$ would have intrinsic sense. So, the above considerations motivate the following

DEFINITION. *The operator* $\Delta = \nabla \circ \mathrm{E} : \bar{\Lambda}^n \to \varkappa(\pi)$ with $\nabla \in \mathcal{C}\,\mathrm{Diff}(\varkappa, \varkappa)$ *is called Hamiltonian if the bracket* $\{\,,\}$ *defined on* L *by*

$$\{\Omega_1, \Omega_2\} = \Delta(\omega_1)(\Omega_2), \tag{2}$$

where $\Omega_1 = [\omega_1] \in L$, *is skew-symmetric and satisfies the Jacobi identity.*

Here $\Delta(\omega_1)(\Omega_2)$ denotes the Lie derivative of $\Omega_2 \in L$ along $\Delta(\omega_1) \in \varkappa(\pi)$. To simplify terminology we will call operator $\nabla : \hat{\varkappa} \to \varkappa$ Hamiltonian as well as $\Delta = \nabla \circ \mathrm{E}$.

Now we intend to prove a criterion for checking an arbitrary operator $\Delta \in \mathcal{C}\,\mathrm{Diff}(\hat{\varkappa}, \varkappa)$ to be Hamiltonian. To perform this we need some general formulae described below.

4. THE GREEN FORMULA AND THE EULER OPERATOR

For $\Delta \in \mathcal{C}\,\mathrm{Diff}(P, Q)$ one can define the adjoint operator $\Delta^* \in \mathcal{C}\,\mathrm{Diff}(\hat{Q}, \hat{P})$. This star operation has the usual properties: (1) $\Delta = \Delta^*$ for $\Delta \in \mathcal{C}\,\mathrm{Diff}_0(\mathscr{F}, \bar{\Lambda}^n)$; (2) $\Delta^* = -\Delta$ for $\Delta \in \mathcal{C}D$; (3) $(\Delta \circ \nabla)^* = \nabla^* \circ \Delta^*$; (4) $(\Delta^*)^* = \Delta$. These imply that $(\Sigma\, a_\sigma D^\sigma)^* = \Sigma(-1)^{|\sigma|} D^\sigma \circ a_\sigma$ and $(\Delta^*)_{ij} = (\Delta_{ji})^*$ for a matrix operator.

If $\Delta \in \mathcal{C}\,\mathrm{Diff}(P, Q)$ then the Green formula holds:

$$\langle \Delta(p), q \rangle - \langle p, \Delta^*(q) \rangle = \bar{d}\text{Ж}(q \circ \Delta \circ p), \quad p \in P, \quad q \in \hat{Q}.$$

Here $\langle , \rangle$ denotes the natural pairing $R \times \hat{R} \to \bar{\Lambda}^n$ for an $\mathscr{F}$-module R, $\text{Ж} : \mathcal{C}\,\mathrm{Diff}(\mathscr{F}, \bar{\Lambda}^n) \to \bar{\Lambda}^{n-1}$ is a $\mathcal{C}$-differential operator defined up to adding an operator of the form $\bar{d} \circ \nu$ with $\nu : \mathcal{C}\,\mathrm{Diff}(\mathscr{F}, \bar{\Lambda}^n) \to \bar{\Lambda}^{n-2}$ being $\mathcal{C}$-differential, and we identify an element $r \in R$ with the homomorphism $r : \mathscr{F} \to R$, $\varphi \mapsto \varphi r$, $\varphi \in \mathscr{F}$.

Next, the universal linearization operator $\ell_p \in \mathcal{C}\,\mathrm{Diff}(\varkappa, P)$ for $p \in P$, P being an $\mathscr{F}$-module, is defined by $\ell_p(f) = \text{Э}_f(p)$. We assume here that Э_f acts on vector-valued functions component-wisely. Now we have the following formula for Euler operator: $\mathrm{E}(\omega) = \ell_\omega^*(1)$, $\omega \in \bar{\Lambda}^n$. Moreover, if Δ is $\mathcal{C}$-differential then

$$E(\Delta(f)) = \ell_f^*(\Delta^*(1)) + \ell^*_{\Delta^*(1)}(f). \tag{3}$$

The next property characterizes the image of E: locally $\varphi \in \operatorname{im} E$ iff $\ell^*_\varphi = \ell_\varphi$. Further, locally $\omega \in \bar{\Lambda}^n$ belongs to $\bar{d}(\bar{\Lambda}^{n-1})$ iff $E(\omega) = 0$. The formulae mentioned in this section are proved in [7], see also [6].

5. THE SKEW-SYMMETRY PROPERTY

Here we shall deduce a property of Δ which yields the skew-symmetry of the bracket $\{\,,\,\}$ defined by (2). We shall write sometimes $\{\omega_1, \omega_2\}$ instead of $\{\Omega_1, \Omega_2\}$ if $\Omega_i = [\omega_i]$ and «$\approx$» to denote equality modulo $\bar{d}\bar{\Lambda}^{n-1}$ in $\bar{\Lambda}^n$.

First of all, by definition $\{\omega, \rho\} = \text{Э}_{\Delta(E\omega)}(\rho) = \ell_\rho(\Delta(E\omega))$, $\omega, \rho \in \bar{\Lambda}^n$, and $\ell_\rho(\Delta(E\omega)) \approx \langle \ell^*_\rho(1), \Delta(E\omega)\rangle = \langle E\rho, \Delta(E\omega)\rangle$ by the Green formula for ℓ_ρ, i.e.

$$\{\omega, \rho\} \approx \langle E\rho, \Delta(E\omega)\rangle. \tag{4}$$

This is the «usual form» of the Poisson bracket, [8], [9]. Applying the Green formula for Δ to the left side of (4) we see that $\{\omega, \rho\} \approx \langle E\omega, \Delta^*(E\rho)\rangle$. Therefore,

$$\{\omega, \rho\} + \{\rho, \omega\} \approx \langle E\omega, (\Delta + \Delta^*)(E\rho)\rangle,$$

and we search when this expression ≈ 0 identically. Its right side may be presented as $\nabla(E\rho)$ with $\nabla = \langle E\omega, (\Delta + \Delta^*)(.)\rangle$. Then $\nabla(E\rho) \approx 0$ iff $E(\nabla(E\rho)) = 0$ (see section 4). Applying (3) to the last expression we have

$$0 = E(\nabla(E\rho)) = \ell^*_{E\rho}(\nabla^*(1)) + \ell^*_{\nabla^*(1)}(E\rho) \quad \text{for all} \quad \rho \in \bar{\Lambda}^n.$$

Obviously, every vector-valued function on $J^\infty(\pi)$ depending only on x may be locally presented as $E(\rho)$ for some $\rho \in \bar{\Lambda}^n$. But $\ell_f = 0$ for such functions. This shows that $\ell^*_{\nabla^*(1)}(f) = 0$ for all f depending only on x. Since $\ell^*_{\nabla^*(1)}$ is $\mathcal{C}$-differential, this implies that $\ell^*_{\nabla^*(1)} = 0$. Therefore $E(\nabla(E\rho)) = \ell^*_{E_\rho}(\nabla^*(1))$. Choosing ρ to be $\frac{1}{2}\sum_i (u^i)^2 dx_1 \wedge \ldots \wedge dx_n$ we see that ℓ_{E_ρ} is the identity operator. Hence $0 = \ell^*_{E_\rho}(\nabla^*(1)) = \nabla^*(1)$.

So, $\nabla^*(1) = (\Delta + \Delta^*)(E\omega) = 0$ for all $\omega \in \bar{\Lambda}^n$. As above we see that $\Delta + \Delta^* = 0$ because $\Delta + \Delta^*$ is a $\mathcal{C}$-differential operator vanishing on all functions of x. So we have proved

PROPOSITION 1. *The bracket* $\{\,,\}_\Delta$ *is skew-symmetric iff* $\Delta + \Delta^* = 0$, i.e. Δ *itself is skew-symmetric.* ■

6. HAMILTONIAN CRITERIA

Now we are able to prove the basic result of the paper. If a $\mathcal{C}$-differential operator Δ maps vector-functions on M into vectorfunctions on $J^k(\pi)$ (noting M itself by $J^{-1}(\pi)$), we say that its filtration is less or equal to k and denote it as $\Phi(\Delta) \leqslant k$.

THEOREM 1. *The skew-symmetric $\mathcal{C}$-differential operator $\Delta : \hat{\varkappa} \to \varkappa$ is Hamiltonian iff*

$$(5) \qquad [\ni_{\Delta\varphi}, \Delta] = \ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi} \qquad \textit{for all} \quad \varphi \in \operatorname{im} E.$$

Moreover, it suffices to verify (5) only for elements $\varphi \in \hat{\varkappa}$ belonging to the image of E and polynomial in x up to ℓ-th order components, $\ell = \deg \Delta + \Phi(\Delta)$.

Proof. In virtue of Proposition 1 we have only to prove that (5) is equivalent to the Jacobi identity for $\{\,,\}_\Delta$. The last assertion of the theorem is true because both sides of (5) are $\mathcal{C}$-differential of order $\leqslant \ell$ as operators acting on $\varphi \in \operatorname{im} E$ and hence are completely determined by their action on the image of π^*_∞.

Further, rewriting the Jacobi identity as $0 = \{\omega, \{\rho, \chi\}\} - \{\rho, \{\omega, \chi\}\} - \{\{\omega, \rho\}, \chi\} = (\Gamma(\omega) \circ \Gamma(\rho) - \Gamma(\rho) \circ \Gamma(\omega) - \Gamma(\{\omega, \rho\}))(\chi)$ with $\Gamma = \Delta \circ E$ we see that it is equivalent to the operator equality $[\Gamma(\omega), \Gamma(\rho)] = \Gamma(\{\omega, \rho\})$ for all ω, $\rho \in \overline{\Lambda}^n$. Identifying elements of $\varkappa$ with evolution derivations and the latters with their generating functions, we shall calculate the generating functions of both sides of the last equality. First, the generating function of $[\Gamma(\omega), \Gamma(\rho)]$ is $\ni_{\Gamma(\omega)}(\Gamma(\rho)) - \ni_{\Gamma(\rho)}(\Gamma(\omega)) = \ell_{\Gamma(\rho)}(\Gamma(\omega)) - \ni_{\Gamma(\rho)}(\Gamma(\omega)) = (\ell_{\Delta(\varphi)} \circ \Delta - \ni_{\Delta(\varphi)} \circ$ $\circ \Delta)(E\omega)$, where $\varphi = E\rho$.

Further, by (3) and (4), $\Gamma(\{\omega, \rho\}) = \Delta(E(\{\omega, \rho\})) = \Delta(E\langle E\rho, \Delta E\omega\rangle) =$ $= \Delta(\ell^*_{E\omega}(\nabla^*(1)) + \ell^*_{\nabla^*(1)}(E\omega))$, where $\nabla = \langle E\rho, \Delta(\,\cdot\,)\rangle$ and $\nabla^*(1) = \Delta^*(E\rho) =$ $= -\Delta(E\rho)$.

Since $\ell^*_{E\omega} = \ell_{E\omega}$ we have

$$\Gamma(\{\omega, \rho\}) = -\Delta(\ell_{E\omega}(\Delta(E\rho)) - \ell^*_{\Delta(E\rho)}(E\omega)) =$$
$$= -(\Delta \circ \ni_{\Delta\varphi} - \Delta \circ \ell^*_{\Delta\varphi})(E\omega).$$

Therefore, $\ell_{\Delta\varphi} \circ \Delta - \ni_{\Delta\varphi} \circ \Delta = -\Delta \circ \ni_{\Delta\varphi} - \Delta \circ \ell^*_{\Delta\varphi}$ as operators on $\operatorname{im} E$ or, equivalently $[\ni_{\Delta\varphi}, \Delta] = \ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi}$. But operators at both the sides of this equality are $\mathcal{C}$-differential. So they coincide completely, not only on $\operatorname{im} E$. ■

Now we shall illustrate the proved theorem at work.

Example. Supposing the fibering π to be linear we consider a differential operator $\nabla : S \to \mathrm{Sec}(\pi)$, $S = \mathrm{Hom}_{C^\infty(M)}(\mathrm{Sec}(\pi), \Lambda^n(M))$. Then the formula $j(f)^* \circ \hat{\nabla} = \nabla \circ j(f)^*$ defines the operator $\hat{\nabla} \in \mathcal{C}\,\mathrm{Diff}(\hat{\varkappa}, \varkappa)$. Evidently, if ∇ is skew-symmetric, then $\hat{\nabla}$ is too. It is easy to show [10] that $[\ni_f, \hat{\nabla}] = 0$ for all f. This leads to the equality $\ell_{\hat{\nabla}(\varphi)} = \hat{\nabla} \circ \ell_\varphi$. So $\ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi} = \Delta \circ (\ell_\varphi - \ell^*_\varphi) \circ \Delta$, if $\Delta = \hat{\nabla}$. But $\ell_\varphi = \ell^*_\varphi$ if $\varphi \in \mathrm{im}\, E$. This shows that every skew-symmetric operator of the form $\hat{\nabla}$ is Hamiltonian (com. [11]). In what follows our main results will be derived from this theorem. For applications we need its coordinate expression.

COROLLARY. *The* $(m \times m)$*-matrix skew-symmetric operator* $\Delta = \| \Delta_{ij} \|$, $\Delta_{ij} = \sum_\sigma A^{ij}_\sigma D^\sigma$, *is Hamiltonian iff*

$$\sum_{l=i}^{m} \sum_{\lambda,\sigma} \left[\binom{\lambda}{\mu - \sigma} D^{\lambda+\sigma-\mu} A^{li}_\sigma \frac{\partial A^{kj}_\nu}{\partial p^l_\lambda} - \binom{\lambda}{\nu - \sigma} D^{\lambda+\sigma-\nu} A^{lj}_\sigma \frac{\partial A^{ki}_\mu}{\partial p^l_\lambda} + \right.$$

$$(6) \qquad \left. + \sum_\tau (-1)^{|\lambda|} \binom{\lambda+\sigma}{\mu} \binom{\lambda+\sigma-\mu}{\lambda-\tau} A^{kl}_\sigma D^{\lambda+\sigma+\tau-\mu-\nu} \frac{\partial A^{ij}_\tau}{\partial p^l_\lambda} \right] = 0$$

for all $1 \leqslant i, j, k \leqslant m$ *and all multi-indices* μ, ν. *Binomial coefficients for multi-indices are defined by* $\binom{\alpha}{\beta} = \binom{\alpha^1}{\beta^1} \cdots \binom{\alpha^n}{\beta^n}$.

It suffices to check (6) *for* μ *and* ν *with* $|\mu + \nu| \leqslant \deg \Delta + \max \left\{ |\tau + \lambda| : \frac{\partial A^{ij}_\tau}{\partial p^l_\lambda} \neq 0 \right\}$. ■

Now we proceed to describe Hamiltonian operators for some small values of m, n and $k = \deg \Delta$. We begin with establishing some relations between $\Phi(\Delta)$ and filtrations of Δ's coefficients. In doing that some facts concerning the $\mathcal{C}$-Hamiltonian formalism will be useful.

7. THE $\mathcal{C}$-HAMILTONIAN FORMALISM

This is a variant of the «usual» Hamiltonian formalism lifted on $J^\infty(\pi)$. Namely, let

$$\mathrm{Smbl}_k(\pi) = \mathcal{C}\,\mathrm{Diff}_k(F, F)/\mathcal{C}\,\mathrm{Diff}_{k-1}(F, F), \ \mathrm{Smbl}(\pi) = \sum_{k \geqslant 0} \mathrm{Smbl}_k(\pi).$$

For $\Delta \in \mathcal{C}\,\mathrm{Diff}_k(\mathcal{F}, \mathcal{F})$ we write $s_k(\Delta)$ for its image in $\mathrm{Smbl}_k(\pi)$. It is easy to see that $s_{k_1+k_2}([\Delta_1, \Delta_2]) = 0$ if $\Delta_i \in \mathcal{C}\,\mathrm{Diff}_{k_i}(\mathcal{F}, \mathcal{F})$. Therefore, the composition of operators induces the commutative multiplication in $\mathrm{Smbl}\,(\pi)$, $\mathrm{Smbl}_k(\pi) \cdot \mathrm{Smbl}_l(\pi) \subset \mathrm{Smbl}_{k+l}(\pi)$. Moreover, $\mathrm{Smbl}\,(\pi)$ is a Lie algebra. The corresponding bracket is defined by $[s_k(\Delta), s_l(\nabla)] = s_{k+l-1}([\Delta, \nabla])$. Evidently, $s_k(\Delta^*) = (-1)^k s_k(\Delta)$.

In local coordinates elements of $\mathrm{Smbl}\,(\pi)$ may be described as polynomials of $\rho_i = s_1(D_i)$, $i = 1, \ldots, n$, with coefficients in $\mathcal{F}(\pi)$, and for $f, g \in \mathrm{Smbl}\,(\pi)$ we have

$$[f, g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \rho_i} D_i(g) - \frac{\partial g}{\partial \rho_i} D_i(f) \right),$$

where it is supposed that $D_i(\rho_j) = 0$. For more details see [1].

8. FILTRATION OF HAMILTONIAN OPERATORS OF THE FIRST ORDER

We suppose here that π is a one-dimensional fibering and $\Delta \in \mathcal{C}\,\mathrm{Diff}_1(\hat{\varkappa}, \varkappa)$ is Hamiltonian. In virtue of its skew-symmetry, Δ has the form $\sum_i \left(f_i D_i + \frac{1}{2} D_i(f_i) \right)$. Below we work in a local chart, identifying $\varkappa$ and $\hat{\varkappa}$ with $\mathcal{F}$.

LEMMA. $\Phi(\Delta)$ *is odd.*

Proof. It follows from definitions that $\deg \ell_{\Delta\varphi} \leqslant \Phi(\Delta)$, if $\varphi \in C^\infty(M)$, and the equality holds for some φ. Further, for such φ we have $s_{a+b}(\ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi}) = s_a(\Delta) \cdot s_b(\ell_{\Delta\varphi} + \ell^*_{\Delta\varphi})$, $a = \deg \Delta$, $b = \Phi(\Delta)$. Therefore, supposing $\Phi(\Delta) \geqslant 1$, for validity of (5), it is necessary that $s_{a+b}(\ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi}) = 0$ because otherwise $\deg [\text{Э}_{\Delta\varphi}, \Delta] \leqslant \deg \Delta < \deg (\ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi})$. So, if $\Phi(\Delta) \geqslant 1$, then $0 = s_b(\ell_{\Delta\varphi} + \ell^*_{\Delta\varphi}) = s_b(\ell_{\Delta\varphi}) + (-1)^b s_b(\ell_{\Delta\varphi})$. The case $\Phi(\Delta) = 0$ is evidently impossible. ■

Remark. The condition $\deg \Delta = 1$ is unessential for this lemma.

PROPOSITION 2. $\Phi(f_i) \leqslant 2$, $i = 1, \ldots, n$.

Proof. We write $\Phi(f) = k$ if $f \in \mathcal{F}_k \setminus \mathcal{F}_{k-1}$. Let $\Delta = X + \alpha/2$, $X = \Sigma f_i D_i$, $\alpha = \Sigma D_i(f_i)$. The next two cases arise now:

(1) (general) $\Phi(\alpha) = \max_i \Phi(f_i) + 1$ and

(2) $\Phi(\alpha) \leqslant \max \Phi(f_i)$.

First, suppose that (1) holds and $\mu = \max \Phi(f_i) > 2$. Then $\Phi(\alpha) = \mu + 1$ and by the lemma μ is even. This shows that $s_{\mu+2}(\ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi}) = 0$, $\varphi \in C^\infty(M)$, or equivalently, that $\deg(\ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi}) \leqslant \mu + 1$. Therefore, because of $\deg [\ni_{\Delta\varphi}, \Delta] = 1$ the equality (5) may hold only if $s_{\mu+1}(\ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi}) = 0$.

If $\varphi = 2\psi \in C^\infty(M)$, then $\ell_{\Delta\varphi} = \psi\, \ell_\alpha + 2\, \ell_{X\psi}$. Substituting it into $\ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi}$ and performing some elementary calculations we obtain

$$\begin{aligned} \ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi} = {} & \psi \cdot (X \circ (\ell_\alpha + \ell^*_\alpha) + [\ell_\alpha, X]) + \\ & + \frac{\alpha}{2}\, \ell^*_\alpha \circ \psi + \psi\, \ell_\alpha \circ \frac{\alpha}{2} + 2(\ell_{X\psi} \circ X + X \circ \ell^*_{X\psi}) + \\ & + X(\psi)\, \ell^*_\alpha + X \circ [\ell^*_\alpha, \psi] + \ell_{X\psi} \circ \alpha + \alpha \circ \ell^*_{X\psi} . \end{aligned} \tag{7}$$

Now, $\left(\psi\, \ell_\alpha \circ \frac{\alpha}{2}\right)^* = \frac{\alpha}{2}\, \ell^*_\alpha \circ \psi$. So, $\deg\left(\frac{\alpha}{2}\, \ell^*_\alpha \circ \psi + \psi \circ \ell_\alpha \circ \frac{\alpha}{2}\right) \leqslant \mu$ because $\deg \ell_\alpha = \mu + 1$ and μ is even. Similarly, $\deg(\ell_\alpha + \ell^*_\alpha) \leqslant \mu$. Also $\deg(\ell_{X\psi} \circ \alpha + \alpha\, \ell^*_{X\psi}) \leqslant \mu$, because $\Phi(X(\psi)) \leqslant \mu$. Using these remarks we obtain

$$\begin{aligned} s_{\mu+1}(\ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi}) = {} & \psi \cdot (s_1(X) s_\mu(\ell_\alpha + \ell^*_\alpha) + \\ & + s_{\mu+1}([\ell_\alpha, X])) + 4 s_1(X)\, s_\mu(\ell_{X\psi}) - X(\psi)\, s_{\mu+1}(\ell_\alpha) - \\ & - s_1(X)\, s_\mu([\ell_\alpha, \psi]) = 0. \end{aligned} \tag{8}$$

Let $s_1(X) = w = \Sigma f_i\, \rho_i$, $s_{\mu+1}(\ell_\alpha) = \upsilon$, $s_\mu(\ell_{f_i}) = \upsilon_i$. Then $\upsilon = \Sigma\, \upsilon_i \rho_i$ and for $\psi = x_s$ we have $s_\mu([\ell_\alpha, \psi]) = \partial \upsilon / \partial \rho_s$.

In these notations (8) may be rewritten as

$$w \cdot s_\mu(\ell_\alpha + \ell^*_\alpha) + [\upsilon, w] = 0, \quad \text{for} \quad \psi = 1,$$

and as $f_s \upsilon + w\, \partial \upsilon / \partial \rho_s = 4\, w \upsilon_s$, for $\psi = x_s$, $s = 1, \ldots, n$. Multiplying these equalities by ρ_s and then summing we get the equality $w \cdot \left(\upsilon + \Sigma \rho_s\, \frac{\partial \upsilon}{\partial \rho_s}\right) = 4\, w \upsilon$ or, equivalently $\Sigma \rho_s\, \frac{\partial \upsilon}{\partial \rho_s} = 3 \upsilon$. Therefore, by «the Euler theorem» υ as a functions on ρ_s is homogeneous of degree 3. Thus $\deg \ell_\alpha \leqslant 3$, i.e. $\Phi(\Delta) \leqslant 3$, $\Phi(f_i) \leqslant 2$.

To finish the proof it suffices to show that the assumptions $\Phi(\alpha) \leqslant \max\limits_i \Phi(f_i) = \mu > 1$ are impossible. If so, μ is odd by the lemma, and

$$\deg\left(\psi\, \ell_\alpha \circ \frac{\alpha}{2} + \frac{\alpha}{2}\, \ell^*_\alpha \circ \psi\right) = \deg\left(\psi\, \ell_\alpha \circ \frac{\alpha}{2} + \left(\psi\, \ell_\alpha \circ \frac{\alpha}{2}\right)^*\right) < \mu.$$

Similarly, $\deg(\ell_{X\psi} \circ \alpha + \alpha \circ \ell^*_{X\psi}) < \mu$. Taking it into account one can deduce from (7) by the direct calculation that $s(\psi) = s_\mu(\ell_{\Delta\varphi} \circ \Delta + \Delta \circ \ell^*_{\Delta\varphi}) = A\psi + {}$ $+ \sum_s \left(C_s \dfrac{\partial\psi}{\partial x_s} + \left[\dfrac{\partial\psi}{\partial x_s}, wv_s \right] \right)$, where A and C_s do not depend on ψ, and $w = {}$ $= s_1(X)$, $v_s = s_\mu(\ell_{f_s})$. Since $s(\psi)$ needs to be zero for all $\psi \in C^\infty(M)$, if $\mu > 1$, it follows that $A = C_s = \Sigma[\partial\psi/\partial x_s, wv_s] = 0$. Substituting $\psi = x_s x_k$ into the last equality we obtain $[wv_s, x_k] + [wv_k, x_s] = \dfrac{\partial(wv_s)}{\partial\rho_k} + \dfrac{\partial(wv_k)}{\partial\rho_s} = 0$ for all s, k and hence $wv_k = \sum_i \alpha^i_k \rho_i$, where α^i_k do not depend on ρ_j, $1 \leqslant j \leqslant n$, and $\alpha^i_k + {}$ $+ \alpha^k_i = 0$. But by assumption not all v_k vanish. Therefore, if $v_k \neq 0$, then wv_k is a non-zero homogeneous polynomial of ρ_j of degree $\mu + 1 > 2$. ■

9. FILTRATION OF THE THIRD-ORDER HAMILTONIAN OPERATOR

In this section we suppose $m = n = 1$.

PROPOSITION 3. *If* $\Delta = f_3 D^3 + f_2 D^2 + f_1 D + f_0$ *is Hamiltonian, then* $f_k \in F_{5-k}$, $k = 0, 1, 2, 3$.

Proof. It is easy to see that every skew-symmetric operator of the third order has the form $\Delta = f_3 D^3 + \dfrac{3}{2} D(f_3) D^2 + f_1 D + \dfrac{1}{2} D(f_1) - \dfrac{1}{4} D^3(f_3)$. Because of proposition 2 we can assume that f_3 has no zeros in the domain of consideration. Let s be the least number with $f_k \in F_{s-k}$. First, suppose that $s \geqslant 8$. Then equations (6) with $\nu = 0, 1, 2, 3$ and $\mu = s + 3 - \nu$ form a linear homogeneous algebraic system with respect to $f_3 \partial f_3/\partial p_{s-3}$ and $f_3 \partial f_1/\partial p_{s-1}$ which is of the rank 2. Therefore, $\partial f_3/\partial p_{s-3} = \partial f_1/\partial p_{s-1} = \partial f_2/\partial p_{s-2} = \partial f_0/\partial p_s = 0$, but this contradicts to the choice of s, and hence $s \leqslant 7$.

Further, all equations (6) with $\mu + \nu = 10$ are proportional to

$$6\,\partial f_1/\partial p_6 - 5\,\partial f_3/\partial p_4 = 0. \tag{9}$$

For $\mu + \nu = 9$ we have a system which may be reduced to

$$38\,\partial f_1/\partial p_5 - 23\,\partial f_3/\partial p_3 - 28 D(\partial f_3/\partial p_4) = 0, \tag{10}$$

$$6 f_3\,\partial f_3/\partial p_3 + 19 D(f_3)\,\partial f_3/\partial p_4 - 14 f_3 D(\partial f_3/\partial p_4) = 0. \tag{11}$$

Similarly, all equations arising from (6) for $\mu + \nu = 8$ may be reduced to

$$
\begin{aligned}
&76f_3\,\partial f_1/\partial p_4 - 76\,\partial f_3/\partial p_2 - \frac{38}{3}\,f_1\,\partial f_3/\partial p_4 - \\
&- 9D(f_3)\,\partial f_3/\partial p_3 - 96f_3D(\partial f_3/\partial p_3) + \\
&+ 40D(f_3)D(\partial f_3/\partial p_4) - 80f_3D^2(\partial f_3/\partial p_4) = 0.
\end{aligned}
\tag{12}
$$

Differentiating (11) in p_5 and (12) in p_6, and substituting in the result $\partial f_3/\partial p_4$ instead of $\partial f_1/\partial p_6$ in accordance with (9) we obtain two linear independent algebraic equations on $f_3\,\partial^2 f_3/(\partial p_4)^2$ and $(\partial f_3/\partial p_4)^2$ yielding $\partial f_3/\partial p_4 = 0$.

Accounting the latter, system (9) – (12) reduces to $\partial f_1/\partial p_4 = \partial f_3/\partial p_2$, $\partial f_1/\partial p_6 = \partial f_1/\partial p_5 = \partial f_3/\partial p_5 = 0$ and as a consequence, $\partial f_2/\partial p_4 = \partial f_0/\partial p_6 = 0$. ■

Remark. The Hamiltonian operator $\Delta = \left(\frac{1}{p_2}D\right)^3 \circ \frac{1}{p_2}$ shows that proposition 3 gives an exact estimate for $\Phi(f_i)$. This also shows the description of the third-order Hamiltonian operators given in [12] to be incomplete.

10. HAMILTONIAN MAPS

To prove «the Darboux lemma» we need the notion of a Hamiltonian map which is analogous to that of a canonical transformation in classical mechanics. The most natural is to introduce it in terms of the category of non-linear differential equations (ND), [6]. For simplicity we consider only ND-coverings for «simple» objects of (ND), namely, for $\mathrm{F}(\pi)$. Because of all considerations being local we often deal with localizations of L, $\varkappa$, $\hat{\varkappa}$ etc. using the same notations for them.

So, let π and π' be fiberings over the bases of the same dimension n, and $F: J^\infty(\pi) \to J^\infty(\pi')$ be a C^∞ map. This means that $f \circ F \in \mathrm{F}(\pi)$ for any $f \in \mathrm{F}(\pi')$. Thus we have the homomorphism $F^*: \mathscr{F}(\pi') \to \mathscr{F}(\pi)$ prolongable to the Grassman algebra homomorphism $F^*: \Lambda(J^\infty(\pi')) \to \Lambda(J^\infty(\pi))$ commuting with the exterior differential d. The map F is called an ND-covering if $F^*(C\Lambda(\pi')) \subset C\Lambda(\pi)$ and at any point $\theta \in J^\infty(\pi)$ the corresponding factor operator $\overline{\Lambda}^n(\pi')|_{F(\theta)} \to \overline{\Lambda}^n(\pi)|_\theta$ has the trivial kernel.

Every ND-covering is uniquely determined by the restriction $F^*|_{\mathrm{F}_0(\pi')}$ or by the corresponding map $F_0: J^k(\pi) \to J^0(\pi')$, and conversely, such a map for which the element $\overline{F_0^*(dx_1 \wedge \ldots \wedge dx_n)} \in \overline{\Lambda}^n(\pi)$ does not vanish, determines an ND-covering F satisfying $F^*|_{\mathrm{F}_0(\pi')} = F_0^*$.

In particular, an ND-covering may be obtained from a diffeomorphism of the fibre spaces or, if $m = 1$, from a contact diffeomorphism of the 1-jet manifolds. ND-coverings of that kind are called *Lie transformations*. Sometimes we shall identify the mentioned difffeomorphisms with the corresponding Lie transformations.

It follows from definitions that, given an ND-covering $F : J^\infty(\pi) \to J^\infty(\pi')$, one can define a linear operator $F^* : \mathrm{L}(\pi') \to \mathrm{L}(\pi)$. Suppose π and π' are equipped with the Poisson brackets $\{\,,\}$ and $\{\,,\}'$, respectively. The ND-covering F is called *Hamiltonian* if $F^*\{h_1, h_2\}' = \{F^*h_1, F^*h_2\}$ for all h_1, $h_2 \in \mathrm{L}(\pi')$. In this case the bracket $\{\,,\}'$ is said to be obtained from $\{\,,\}$ by F.

Note that a Lie transformation F determines in a natural way module isomorphisms $F_* : \varkappa(\pi) \to \varkappa(\pi')$ and $F^* : \hat{\varkappa}(\pi') \to \hat{\varkappa}(\pi)$ over the algebra homomorphism $F^* : \mathrm{F}(\pi') \to \mathrm{F}(\pi)$. So the property of being Hamiltonian may be formulated in terms of the corresponding Hamiltonian operators:

$$\Delta' = F_* \circ \Delta \circ F^*.$$

For example, the Hamiltonian operator $\left(\frac{1}{p_2} D\right)^3 \circ \frac{1}{p_2}$ in the previous section may be obtained from D^3 by the contact transformation $(x, u, p) \mapsto (p, u - px, -x)$.

Generally an ND-covering does not induce any natural map of «vetor fields». On the contrary, there exists the natural opeator $F^* : \hat{\varkappa}(\pi') \to \hat{\varkappa}(\pi)$. It is due to the fact that every form $\omega \in \Lambda^{n+1}$ generates an element $[\omega] = \Delta^*_\omega(1) \in \hat{\varkappa}$, where $\Delta_\omega \in \mathcal{C}\,\mathrm{Diff}(\varkappa, \bar{\Lambda}^n)$ is defined by $\Delta_\omega\ (X \bmod \mathcal{C}D) = X \lrcorner\, \omega \bmod \mathcal{C}\Lambda^n$. If ω_1, $\omega_2 \in \Lambda^{n+1}$ generate the same element of $\hat{\varkappa}$, then it holds also for $F^*(\omega_1)$ and $F^*(\omega_2)$, F being an ND-covering [13]. For Lie transformations this definition of $F^* : \hat{\varkappa}(\pi') \to \hat{\varkappa}(\pi)$ coincides with the above.

Now, the ND-covering F is Hamiltonian iff $F^*\langle \Delta'\alpha, \beta\rangle = \langle \Delta F^*\alpha, F^*\beta\rangle$ for all α, $\beta \in \hat{\varkappa}(\pi')$. We emphasize that, in general, $F^* : \hat{\varkappa}(\pi') \to \hat{\varkappa}(\pi)$ is not a module homomorphism and therefore $\deg \Delta'$ doesn't need to be equal to $\deg \Delta$, Δ' being obtained from Δ by an ND-covering. Moreover, the existence of Δ' for arbitrary Δ and F is not guaranteed. Now we proceed to establish normal forms of Hamiltonian operators under Lie transformations.

11. THE CASE m = n = K = 1

The operator $\Delta = f_1 D + f_0$ is said to be non-degenerate if f_1 nowhere vanishes in its domain.

THEOREM 2. *Let $m = n = 1$. Then any two non-degenerate Hamiltonian operators of the first order are locally equivalent up to the sign under the Lie transformations.*

Proof. In the case accordingly to proposition 2, the Hamiltonian operators have the form $\Delta = fD + \frac{1}{2} D(f)$, with $f \in F_2$. Because of non-degeneracy we may assume f to be positive. We shall prove existence of a Lie transformation transforming Δ into D.

By direct calculations we obtain that system (6) for Δ is equivalent to

$$3D(f) \cdot f_w - 2f \cdot D(f_w) + 2f \cdot f_p = 0,$$

where $w = p_2$, which may be reduced to

$$D(Q_w) - Q_p = 0 \tag{13}$$

by the substitution $f = Q^{-2}$.

Differentiating (13) in p_3 gives $Q_{ww} = 0$, which yields $Q = aw + b$, with a, $b \in F_1$.

Now, let Φ be the set of all not vanishing functions $\varphi \in F_1(\pi)$ such that the operator $\frac{1}{\varphi} \circ \Delta \circ \frac{1}{\varphi}$ is Hamiltonian as well as Δ. Inserting φQ into (13) we obtain that $\varphi \in \Phi$ iff $X\varphi = 0$, where $X = a\, \partial/\partial x + pa\, \partial/\partial u - b\, \partial/\partial p$ is a vector field on $J^1(\pi)$. Obviously, $X \lrcorner (du - pdx) = 0$.

Now we use an elementary fact from contact geometry leaving the reader to prove it.

LEMMA. *Let X be a vector field on $J^1(\pi)$ and $X \lrcorner (du - pdx) = 0$. Then in a neighbourhood of any point where X does not vanish, there exist contact coordinates $(\tilde{x}, \tilde{u}, \tilde{p})$ such that $X = \partial/\partial\tilde{p}$.*

In these coordinates the function $\tilde{Q}$, determining operator Δ, does not depend on w. Moreover, in view of (13), $\tilde{Q} \in F_0$. Therefore, this contact transformation reduces Δ to $f(x,u)D + \frac{1}{2} D(f)$. Finally, straightforward calculations show that the diffeomorphism $(x, u) \mapsto (x, \int f^{-1/2} du)$ of $J^0(\pi)$ transforms Δ into D. ∎

Remark. Two Hamiltonian operators D and $-D$ are not equivalent via contact trasformations.

12. THE CASE m = n = 1, K = 3

The operator $\Delta = f_3 D^3 + f_2 D^2 + f_1 D + f_0$ is called non-degenerate if f_3 nowhere vanishes in its domain.

THEOREM 3. *Locally any non-degenerate Hamiltonian operator of the third order,* $m = n = 1$, *may be transformed into an operator of the form:*

$$\pm D^3 + 2\lambda u\, D + \lambda p,$$

by a Lie transformation, λ *being a non-negative constant. This form is unique.*

Proof. Eqs. (6) for $6 \leqslant \mu + \nu \leqslant 8$ and the proposition 3 imply that the Hamiltonian operators have the form:

$$\Delta = f_3 D^3 + f_2 D^2 + f_1 D + f_0,$$

where $f_k \in \mathcal{F}_{5-k}$ and f_3 satisfies

$$5(\partial f_3/\partial p_2)^2 = 4 f_3\, \partial^2 f_3/(\partial p_2)^2.$$

Supposing $f_3 > 0$, we obtain $f_3 = (ap_2 + b)^{-4}$, a, $b \in \mathcal{F}_1$. It is easy to see that there exists a function $\varphi \in \mathcal{F}_1$ such that $Q = \varphi \cdot (ap_2 + b)$ satisfies (13). So Δ has the same symbol as the composition operator $\hat{\varkappa} \xrightarrow{\Delta_1} \varkappa \to \mathcal{F} \xrightarrow{\bar{d}} \overline{\Lambda}^1 \to \hat{\varkappa} \xrightarrow{\Delta_1} \varkappa$, where $\Delta_1 = \dfrac{1}{Q} \circ D \circ \dfrac{1}{Q}$ is a Hamiltonian operator of the first order and the homomorphisms $\varkappa \to \mathcal{F}$, $\overline{\Lambda}^1 \to \hat{\varkappa}$ are inverse to those determined by $\zeta = \ni_{\psi}$ mod $CD \in \varkappa$. $\psi = \varphi^{-2} \in \mathcal{F}_1$. In virtue of the Theorem 2, Δ_1 may be transformed by a Lie transformation into D, while ζ would preserve its form with another function $\psi \in \mathcal{F}_1$ since this form is equivalent to the existence of a contact field $X \in \zeta$. So, the operator Δ after the transformation will have a leading coefficient (still denoted f_3) belonging to $\mathcal{F}_1$. Now, equations (6) for $5 \leqslant \mu + \nu \leqslant 7$ imply

$$2 f_3 \cdot \partial^2 f_3/(\partial p)^2 = 3(\partial f_3/\partial p)^2,$$

which yields $f_3 = (\alpha p + \beta)^{-2}$, α, $\beta \in \mathcal{F}_0$. Choosing a function $\varphi \in \mathcal{F}_0$ such that $d(\varphi\alpha\, du + \varphi\beta\, dx) = 0$, we obtain: $\varphi \cdot (\alpha p + \beta)\, \bar{d}x = \varphi\alpha\, \bar{d}u + \varphi\beta\, \bar{d}x = \bar{d}\psi$ for some $\psi \in \mathcal{F}_0$. Therefore, f_3 coincides with the leading coefficient of the composition operator

$$\hat{\varkappa} \to \mathcal{F} \xrightarrow{\bar{d}} \overline{\Lambda}^1 \to \mathcal{F} \xrightarrow{\bar{d}} \overline{\Lambda}^1 \to \mathcal{F} \xrightarrow{\bar{d}} \overline{\Lambda}^1 \to \varkappa,$$

where the two homomorphisms $\overline{\Lambda}^1 \to \mathcal{F}$ are inverse to those determined by $\bar{d}\psi \in \overline{\Lambda}^1$, while $\hat{\varkappa} \to \mathcal{F}$ and $\overline{\Lambda}^1 \to \varkappa$ are the inverses of homomorphisms determined by $\left[\dfrac{1}{\varphi}\, du \wedge dx\right] \in \hat{\varkappa}$. Since φ, $\psi \in \mathcal{F}_0$, there is a diffeomorphism of $J^0(\pi)$, transforming $d\psi$ into dx and $\dfrac{1}{\varphi}\, du \wedge dx$ into $du \wedge dx$. So, by a Lie transformation Δ may be transformed into an operator with a leading coefficient equal to 1. Now the skew-symmetry implies the second coefficient of such an operator to be equal to zero and the equations (6) for f_1 and f_0 look as

$$D(\partial f_1/\partial u) = 0,$$

$$\partial f_1/\partial p = \partial f_1/\partial p_2 = \partial f_1/\partial p_3 = \partial f_1/\partial p_4 = 0,$$

$$f_0 = \frac{1}{2} D(f_1).$$

This yields $f_1 = \lambda u + \varphi$, where λ is a constant and φ is a function on x.

By straightforward calculating we obtain that a contact transformation $(x, u, p) \mapsto (y, v, q)$ preserves the leading coefficient of the Hamiltonian operator to equal 1 iff

$$\begin{cases} \partial y/\partial p = \partial y/\partial u = \partial v/\partial p = 0, \\ \partial v/\partial u\,(\partial y/\partial x)^2 = \pm 1, \end{cases}$$

and therefore $v = \pm u\, \xi^{-2} + \psi$, where ψ and $\xi = \partial y/\partial x$ depend only on x. In this case Δ transforms into $D^3 + f_1 D + \frac{1}{2} D(f_1)$ with

$$f_1 = \pm \lambda u + 2\xi''/\xi - 3(\xi'/\xi)^2 + \xi^2 \cdot (\lambda \psi + \varphi(y)).$$

This shows that we can only change the λ's sign and get rid of the summand φ via the Lie transformations. ■

13. THE CASE m = n = 1, K = 5

Calculations in this case are similar to the previous ones, but essentially more cumbersome. So, we indicate here only main steps.

The operator

$$\Delta = f_5 D^5 + f_4 D^4 + f_3 D^3 + f_2 D^2 + f_1 D + f_0$$

is called non-degenerate if f_5 nowhere vanishes in its domain.

First, as in sec. 9, eqs. (6) with $\mu + \nu \geqslant 8$ imply

PROPOSITION 4. *If Δ is a Hamiltonian operator with coefficients f_r, $r = 0, 1, \ldots, 5$, then $f_r \in \mathcal{F}_{7-r}$.*

Next, this proposition and eqs. (6) with $7 \leqslant \mu + \nu \leqslant 12$ imply that $f_5 \in \mathcal{F}_2$ satisfies

$$6 f_5\, \partial^2 f_5/(\partial p_2)^2 = 7(\partial f_5/\partial p_2)^2.$$

By solving this equation and supposing $f_5 > 0$ one may obtain that Δ has the same symbol as the composition of the following operators:

$$\hat{\varkappa} \xrightarrow{\Delta_1} \varkappa \to \mathcal{F} \xrightarrow{\bar{d}} \overline{\Lambda}^1 \to \hat{\varkappa} \xrightarrow{\Delta_1} \varkappa \to \mathcal{F} \xrightarrow{\bar{d}} \overline{\Lambda}^1 \to \hat{\varkappa} \xrightarrow{\Delta_1} \varkappa .$$

Here Δ_1 is a Hamiltonian operator of the first order and homomorphisms $\varkappa \to \mathcal{F}$, $\overline{\Lambda}^1 \to \hat{\varkappa}$ are inverse to those determined by an element $\xi \in \varkappa$ which is the equivalence class of a contact field on $J^1(\pi)$. Hence, by Theorem 2, Δ may be transformed by a Lie transformation into an operator with a leading coefficient belonging to $\mathcal{F}_1$. Eqs. (6) with $5 \leqslant \mu + \nu \leqslant 11$ yield for that coefficient:

$$4 f_5 \partial^2 f_5 / (\partial p_1)^2 = 5 (\partial f_5 / \partial p_1)^2 .$$

This implies that f_5 coincides with the leading coefficient of the composition operator

$$\hat{\varkappa} \to \mathcal{F} \xrightarrow{\bar{d}} \overline{\Lambda}^1 \to \mathcal{F} \xrightarrow{\bar{d}} \overline{\Lambda}^1 \to \mathcal{F} \xrightarrow{\bar{d}} \overline{\Lambda}^1 \to \mathcal{F} \xrightarrow{\bar{d}} \overline{\Lambda}^1 \to \mathcal{F} \xrightarrow{\bar{d}} \overline{\Lambda}^1 \to \varkappa ,$$

where included four homomorphisms $\overline{\Lambda}^1 \to \mathcal{F}$ are inverse to those determined by $\bar{d}\psi \in \overline{\Lambda}^1$, while $\hat{\varkappa} \to \mathcal{F}$ and $\overline{\Lambda}^1 \to \varkappa$ are the inverses of homomorphisms determined by $[\varphi\, du \wedge dx] \in \hat{\varkappa}$, where φ, $\psi \in \mathcal{F}_0$. So, there exists a Lie transformation determined by a diffeomorphism of $J^0(\pi)$ and making the leading coefficient of Δ to be equal to 1. Then by Δ's skew-symmetry, $f_4 = 0$, and after some manipulations with eqs. (6), $5 \leqslant \mu + \nu \leqslant 10$, one may obtain that in this case $f_r \in \mathcal{F}_{-1}$, $r = 0, 1, 2, 3$. One may also get rid of the D^3's coefficient by a diffeomorphisms of the base M. Then D^2's one vanishes too. So we have

THEOREM 4. *Let* $m = n = 1$. *Then any non-degenerate Hamiltonian operator of the fifth order may be transformed locally into an operator of the form*

$$\pm D^5 + 2\varphi D + \frac{d\varphi}{dx} ,$$

with $\varphi \in \mathcal{F}_{-1}$.

Remark. This form is not unique.

14. THE CASE m = K = 1, n = 2

In this section we write x, y, D_x, D_y instead of x_1, x_2, D_1, D_2, and omit brackets and commas in writing multi-indices. The operator $\Delta = f_1 D_x + f_2 D_y + f_0$ is said to be non-degenerate if $f_1^2 + f_2^2$ nowhere vanishes in its domain.

THEOREM 5. *Let* $m = 1$, $n = 2$. *Then any two non-degenerate first-order Hamil* *tanian operators are locally equivalent under the Lie transformations.*

Proof. This theorem has a rather cumbersome proof, so we drop a lot of details.

In the case the Proposition 2 shows that the Hamiltonian operators have the form $\Delta = f_1 D_x + f_2 D_y + \frac{1}{2} D_x(f_1) + \frac{1}{2} D_y(f_2)$, $f_1, f_2 \in F_2$. Because of its non-degeneracy and performing a Lie transformation, if needed, we may assume that neither f_1 nor f_2 vanish. We shall prove that there is a Lie transformation trasforming Δ into D_x.

Excluding in equations (6) for $3 \leqslant |\mu + \nu| \leqslant 4$ the partial derivatives of f_1 and f_2 in p_{11}, one can obtain

$$f_1^{-2} \cdot \left(\frac{\partial^2 f_1}{\partial p_{20}\, \partial p_{02}} \cdot f_1 - \frac{\partial f_1}{\partial p_{20}} \cdot \frac{\partial f_1}{\partial p_{02}} \right) =$$

$$= f_2^{-2} \cdot \left(\frac{\partial^2 f_2}{\partial p_{20}\, \partial p_{02}} \cdot f_2 - \frac{\partial f_2}{\partial p_{20}} \cdot \frac{\partial f_2}{\partial p_{02}} \right),$$

which is equivalent to

$$\frac{\partial^2}{\partial p_{20}\, \partial p_{02}} \ln \left| \frac{f_1}{f_2} \right| = 0.$$

Hence, f_1 and f_2 may be written in the form: $f_1 = \eta Q^{-2}$, $f_2 = -\xi Q^{-2}$, with $Q, \xi, \eta \in F_2$, satisfying $\dfrac{\partial \eta}{\partial p_{20}} = \dfrac{\partial \xi}{\partial p_{02}} = 0$.

Inserting these into equations (6) with the same μ and ν gives:

$$\frac{\partial^2 \xi}{(\partial p_{20})^2} = \frac{\partial^2 \eta}{(\partial p_{02})^2} = \frac{\partial^2 Q}{(\partial p_{20})^2} = \frac{\partial^2 Q}{(\partial p_{02})^2} = 0,$$

$$\left(\frac{\partial \xi}{\partial p_{20}} - \frac{\partial \eta}{\partial p_{11}} \right) \cdot \frac{1}{\eta} = \left(\frac{\partial \eta}{\partial p_{02}} - \frac{\partial \xi}{\partial p_{11}} \right) \cdot \frac{1}{\xi}.$$

Comparing the two we see that both sides of the last equation are independent of p_{20} and p_{02}. So, there is a positive function $\lambda \in F_2$ also independent of p_{20} and p_{02}, which satisfies

$$\frac{1}{\lambda} \frac{\partial \lambda}{\partial p_{11}} = \left(\frac{\partial \xi}{\partial p_{20}} - \frac{\partial \eta}{\partial p_{11}} \right) \cdot \frac{1}{\eta} = \left(\frac{\partial \eta}{\partial p_{02}} - \frac{\partial \xi}{\partial p_{11}} \right) \cdot \frac{1}{\xi}.$$

Changing notations, let $\lambda\xi$, $\lambda\eta$, $Q\sqrt{\lambda}$ be the new functions ξ, η, Q. Then the above properties still hold, but due to the choice of λ we have also $\partial\xi/\partial p_{11} = \partial\eta/\partial p_{02}$, $\partial\xi/\partial p_{20} = \partial\eta/\partial p_{11}$, and therefore:

$$\frac{\partial^2\xi}{(\partial p_{11})^2} = \frac{\partial^2\xi}{\partial p_{20}\,\partial p_{11}} = \frac{\partial^2\eta}{(\partial p_{11})^2} = \frac{\partial^2\eta}{\partial p_{11}\,\partial p_{02}} = 0,$$

$$\frac{\partial^2 Q}{\partial p_{20}\,\partial p_{11}} = \frac{\partial^2 Q}{\partial p_{11}\,\partial p_{02}} = \frac{\partial^2 Q}{(\partial p_{11})^2} + 2\,\frac{\partial^2 Q}{\partial p_{20}\,\partial p_{02}} = 0.$$

This gives the explicit form of dependence of ξ, η, Q on p_σ, $|\sigma| = 2$:

$$\xi = \alpha p_{20} + \beta p_{11} + \gamma_1, \qquad \eta = \alpha p_{11} + \beta p_{02} + \gamma_2,$$

$$Q = H\cdot(p_{11}^2 - p_{20}p_{02}) + h_1 p_{20} + 2h_2 p_{11} + h_3 p_{02} + h_4,$$

where all new functions are in F_1 and satisfy the following equations by (6):

$$\begin{aligned}
- H\gamma_2 - h_1\beta + h_2\alpha &= 0\\
H\gamma_1 \qquad - h_2\beta + h_3\alpha &= 0\\
h_1\gamma_1 + h_2\gamma_2 \qquad - h_4\alpha &= 0\\
-h_2\gamma_1 - h_3\gamma_2 + h_4\beta \qquad &= 0.
\end{aligned}$$

Since it is a homogeneous linear system with respect to α, β, γ_1, γ_2, which are not all zero

$$\det\begin{pmatrix} 0 & -H & -h_1 & h_2\\ H & 0 & -h_2 & h_3\\ h_1 & h_2 & 0 & -h_4\\ -h_2 & -h_3 & h_4 & 0 \end{pmatrix} = (h_2^2 - h_1h_3 - Hh_4)^2 = 0. \tag{14}$$

Next, equations (6) for $|\mu + \nu| = 3$ have two consequences which we write in the form:

$$\begin{aligned}
&\frac{\partial h_1'}{\partial x} + p_{10}\frac{\partial h_1'}{\partial u} + \frac{\partial h_2'}{\partial y} + p_{01}\frac{\partial h_2'}{\partial u} + \frac{\partial h_1'}{\partial p_{10}}\,h_3' -\\
&\qquad - h_2'\frac{\partial h_2'}{\partial p_{10}} - \frac{\partial h_1'}{\partial p_{01}}\cdot h_2' + h_1'\frac{\partial h_2'}{\partial p_{01}} = 0,
\end{aligned} \tag{15}$$

$$\frac{\partial h_2'}{\partial x} + p_{10}\frac{\partial h_2'}{\partial u} + \frac{\partial h_3'}{\partial y} + p_{01}\frac{\partial h_3'}{\partial u} + h_1'\frac{\partial h_3'}{\partial p_{01}} - \tag{16}$$

$$(16) \qquad -h_2' \frac{\partial h_2'}{\partial p_{01}} + \frac{\partial h_2'}{\partial p_{10}} h_3' - h_2' \cdot \frac{\partial h_2'}{\partial p_{10}} = 0,$$

with $h_k' = \dfrac{h_k}{H}$, $k = 1, 2, 3$ (we assume $H \neq 0$, otherwise it can be achieved by a Lie transformation). Note that Δ has the same leading coefficients as the composition operator

$$\hat{\varkappa} \to \mathcal{F} \to \overline{\Lambda}^1 \xrightarrow{\bar{d}} \overline{\Lambda}^2 \longrightarrow \varkappa,$$

where $\hat{\varkappa} \to \mathcal{F}$ and $\overline{\Lambda}^2 \to \varkappa$ are the inverses of the homomorphisms determined by $[Qdu \wedge dx \wedge dy] \in \hat{\varkappa}$ and the homomorphism $\mathcal{F} \to \overline{\Lambda}^1$ is determined by $\xi \bar{d}x + \eta \bar{d}y \in \overline{\Lambda}^1$.

It is not difficult to show that the established form of dependence of ξ, η, Q on p_σ, $|\sigma| = 2$, is equivalent to the existence of forms belonging to $\Lambda(J^1(\pi))$ and generating the same homomorphisms. Moreover, we shall prove the following

LEMMA. *Let H, $h_1, \ldots, h_4 \in \mathcal{F}_1$ satisfy* (14) - (16). *There exists a positive function $H' \in \mathcal{F}_1$ such that the element of $\hat{\varkappa}$ generated by the form $Q'du \wedge dx \wedge dy \in \Lambda^3$ with*

$$Q' = H'(p_{11}^2 - p_{20}p_{02}) + \frac{H'}{H} h_1 p_{20} + 2\frac{H'}{H} h_2 p_{11} + \frac{H'}{H} h_3 p_{02} + \frac{H'}{H} h_4,$$

belongs to im $\mathcal{E}$.

Proof. The assertion is equivalent to $\ell_{Q'}^* = \ell_{Q'}$, which after expanding reduces to systems of four differential equations. Two of them in view of (14) - (16) appear to be linear combinations of two others, which may be written as

$$X_1(H') = g_1 H', \qquad X_2(H') = g_2 H',$$

where X_1 and X_2 are linearly independent vector fields and g_1, g_2 are functions of h_k', $k = 1, 2, 3$. Equations (15), (16) imply $[X_1, X_2] = 0$, $X_1 g_2 - X_2 g_1 = 0$. This is sufficient for the existence of H'. ■

Further, changing notations once more, we way assume that the form $Qdu \wedge \wedge dx \wedge dy$ generates an element of im $\mathcal{E}$. Consider the set Φ of all nowhere vanishing functions $\varphi \in \mathcal{F}_1$ such that $\varphi Qdu \wedge dx \wedge dy$ also generates an element of im $\mathcal{E}$. Expanding $\ell_{\varphi Q} = \ell_{\varphi Q}^*$ and dealing with Φ as in the section 11, we obtain that there is a Lie transformation transforming Δ into an operator having the same leading coefficients as

$$\hat{\varkappa} \to \mathcal{F} \to \bar{\Lambda}^1 \xrightarrow{\bar{d}} \bar{\Lambda}^2 \to \varkappa,$$

and determined by $[Q\, du \wedge dx \wedge dy] \in \hat{\varkappa}$ and $\xi \bar{d}x + \eta \bar{d}y \in \bar{\Lambda}^1$ with $Q, \xi, \eta \in \mathcal{F}_1$. So, the Lie transformation puts the leading coefficients of Δ into $\mathcal{F}_1$ and reduces equations (6) for $|\mu + \nu| \geqslant 3$ to

$$\partial f_1/\partial p_{10} = \partial f_2/\partial p_{01} = \partial f_1/\partial p_{01} + \partial f_2/\partial p_{10} = 0.$$

Thus, $f_1 = ap_{01} + b_1$, $f_2 = -(ap_{10} + b_2)$ with $a, b_1, b_2 \in \mathcal{F}_0$. So, in the above decomposition one may assume $\hat{\varkappa} \to \mathcal{F}$ and $\Lambda^2 \to \varkappa$ to be the inverses of the homomorphisms determined by $[du \wedge dx \wedge dy] \in \hat{\varkappa}$ and also $\mathcal{F} \to \bar{\Lambda}^1$ to be determined by $a\bar{d}u + b_2\bar{d}x + b_1\bar{d}y \in \bar{\Lambda}^1$, the form $\omega = adu + b_2dx + b_1dy$ belonging to $\Lambda^1(J^0(\pi))$. Now all equations (6) for $|\mu + \nu| \leqslant 2$ reduce to one, which may be written in terms of ω as $\omega \wedge \mathrm{d}\omega = 0$. This implies that in some coordinates on $J^0(\pi)$ the form ω has the form fdy, where $f(x, y, u)$ is a positive function and therefore by a Lie transformation Δ can be transformed into an operator with the same leading coefficients as the above composition operator generated by $[f^{-1/2} du \wedge dx \wedge dy] \in \hat{\varkappa}$ and $\bar{d}y \in \bar{\Lambda}^1$. Finally, performing a diffeomorphism of $J^0(\pi)$ transforming $f^{-1/2} du \wedge dx \wedge dy$ into $du \wedge dx \wedge dy$ and not changing dy, we transform the operator Δ into D_x.

15. THE CASE K = 0

In this section we demonstrate that the theory of the zero-order Hamiltonian operators isn't so trivial as it appears at first glance. Obviously, such an operator is non-degenerate when it is an isomorphism.

THEOREM 6. *The isomorphism* $\Delta : \hat{\varkappa} \to \varkappa$ *is Hamiltonian iff there is a closed form* $\omega \in \Lambda^{n+2}(J^0(\pi))$ *such that* Δ^{-1} *maps* X mod $\mathcal{C}D$ *into the element of* $\hat{\varkappa}$, *generated by* $X \lrcorner\, \omega \in \Lambda^{n+1}$.

Proof. Equations (6) for the zero-order operator reduce to $\Delta^{ij} \in \mathcal{F}_1$ and also to two equations. It will be convenient to write these in terms of $\Omega = \Delta^{-1}$:

$$\frac{\partial \Omega^{ij}}{\partial p^k_\sigma} = -\frac{\partial \Omega^{kj}}{\partial p^i_\sigma}, \qquad 1 \leqslant i, j, k \leqslant m, \qquad |\sigma| = 1, \tag{17}$$

$$\frac{\partial \Omega^{ij}}{\partial u^k} + \frac{\partial \Omega^{jk}}{\partial u^i} + \frac{\partial \Omega^{ki}}{\partial u^j} + \sum_{s=1}^{n} \left(\frac{\partial^2 \Omega^{ij}}{\partial x_s \partial p^k_{\epsilon(s)}} + \sum_{l=1}^{m} p^l_{\epsilon(s)} \frac{\partial^2 \Omega^{ij}}{\partial u^l \partial p^k_{\epsilon(s)}} \right) = 0. \tag{18}$$

Taking into account the skew-symmetry of Δ and therefore of Ω equation (17) means that for any multi-index σ, $|\sigma| = 1$, the expression $\partial^2\Omega^{ij}/\partial p^k_\sigma$ is skew-symmetric in i, j, k. This implies immediately that

$$\frac{\partial^2\Omega^{ij}}{\partial p^k_\sigma\,\partial p^l_\tau} = -\frac{\partial^2\Omega^{ij}}{\partial p^k_\tau\,\partial p^l_\sigma}, \qquad |\sigma| = |\tau| = 1$$

and, in particular,

$$\frac{\partial^2\Omega^{ij}}{\partial p^k_\sigma\,\partial p^l_\sigma} = \frac{\partial^2\Omega^{ij}}{\partial p^k_\sigma\,\partial p^k_\tau} = 0.$$

So, Ω^{ij} is a polynomial of p^k_σ, $|\sigma| = 1$, with coefficients in F_0. Moreover, the F_0-module of all skew-symmetric homomorphisms $\Omega : \varkappa \to \hat{\varkappa}$ satisfying (17) has the following basis. Any pair of index collections:

$$1 \leqslant i_1 < \ldots < i_{r+2} \leqslant m$$
$$1 \leqslant s_1 < \ldots < s_r \leqslant n,$$

where $0 \leqslant r \leqslant \min\{n, m - 2\}$, determines the basis homomorphism:

$$\sum_q (-1)^q\, p^{i_{q(1)}}_{\epsilon(s_1)} \ldots p^{i_{q(r)}}_{\epsilon(s_r)}\, du^{i_{q(r+1)}} \otimes [du^{i_{q(r+2)}} \wedge dx_1 \wedge \ldots \wedge dx_n],$$

(the sum is over all permutations of the set $\{1, \ldots, r + 2\}$). On the other hand, the F_0-module $\Lambda^{n+2}(E)$ has a basis, consisting of forms

$$\frac{\partial}{\partial x_{s_1}} \lrcorner \ldots \lrcorner \frac{\partial}{\partial x_{s_r}} \lrcorner (du^{i_1} \wedge \ldots \wedge du^{i_{r+2}} \wedge dx_1 \wedge \ldots \wedge dx_n),$$

determines by the same pairs.

By straightforward calculations one can prove that such a form determines up to the sign, the above basis homomorphism.

Equation (18) can be guessed as the closure condition for the form ω which determines Ω. There is a rigorous coordinate-free proof of this which needs, however, additional notes and facts, and is, therefore, omitted. ■

It follows from the proof that different closed forms in $\Lambda^{n+2}(E)$ determine different Hamiltonian operators. Therefore, two zero-order non-degenerate Hamiltonian operators are equivalent under the Lie transformations iff their $(n + 2)$-forms are equivalent under the diffeomorphisms of E. Since any two volume forms on a manifold are locally equivalent, the following proposition is obvious:

PROPOSITION 5. *Let $m = 2$. Then any two zero-order non-degenerate Hamiltonian operators are locally equivalent under the Lie transformations.*

The condition $m = 2$ is essential. For example, in the case $m = 4$, $n = 1$, the following three forms are non-equivalent to each other:

$$\omega_1 = (du^1 \wedge du^2 + du^3 \wedge du^4) \wedge dx,$$

$$\omega_2 = (du^1 \wedge du^2 + du^3 \wedge du^4) \wedge (dx + u^1 du^3),$$

$$\omega_3 = (du^1 \wedge du^2 + du^3 \wedge du^4) \wedge (dx + u^1 du^2 + u^2 du^4).$$

The corresponding Hamiltonian operators look as

$$\Delta_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$\Delta_2 = \begin{pmatrix} 0 & -1-u^1p^3 & u^1p^2 & 0 \\ 1+u^1p^3 & 0 & -u^1p^1 & 0 \\ -u^1p^2 & u^1p^1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{-1},$$

$$\Delta_3 = \begin{pmatrix} 0 & -1-u^1p^3-u^2p^4 & u^1p^2 & u^2p^2 \\ 1+u^1p^3+u^2p^4 & 0 & -u^1p^1 & -u^2p^1 \\ -u^1p^2 & u^1p^1 & 0 & -1 \\ -u^2p^2 & u^2p^1 & 1 & 0 \end{pmatrix}^{-1}.$$

16. «THE DARBOUX LEMMA»

The results of sections 12, 13, 15 show the set of Lie transformations to be insufficient for the hypothetical «general Darboux lemma in field theory». The following straightforward results give another candidate, namely, the set of the ND-coverings.

PROPOSITION 6. [14]. *Let $m = n = 1$. Then the map $F: J^1(\pi) \to J^0(\pi)$ defined* *by* $F(x,u,p) = (x, \lambda(u)^2 - p)$, λ *being a constant, determines an ND-covering, mapping the Hamiltonian operator* D *into the Hamiltonian operator* $-D^3 + 4\lambda u D + 2\lambda p$.

PROPOSITION 7. *Let* $m = 2$, $m' = n = n' = 1$, *and* $F : J^1(\pi) \to J^0(\pi')$ *is the map:* $(x, u^1, u^2, p^1, p^2) \mapsto (x, u^1 + \lambda p^2)$, λ *being a constant. Then* F *determines an ND-covering, mapping the Hamiltonian operator* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *into the Hamiltonian operator* $2\lambda D$.

Remark. It holds also for $n = n' > 1$.

PROPOSITION 8. *Let* $m = n = 1$. *Then the map* $F : J^2(\pi) \to J^0(\pi)$ *defined by*

$$F(x,u,p_1,p_2) = (x, p_2 + f \cdot p_1 + (f^2/2 + 2df/dx)\cdot u)$$

with $f \in \mathcal{F}_{-1}$, *determines an ND-covering, mapping the Hamiltonian operator* D *into the Hamiltonian operator*

$$D^5 + \varphi D + \frac{1}{2}\, d\varphi/dx,$$

where

$$\varphi = 2d^3f/dx^3 + 3(df/dx)^2 - 4fd^2f/dx^2 + 3f^2df/dx + f^4/4.$$

Now we can formulate the immediate corollary which is just «the Darboux lemma» for the special cases having been considered above.

THEOREM 6. *Let numbers* m, n, K *be equal to ones of theorems* 2, 3, 4, 5 *or the proposition* 5. *In this case any non-degenerate Hamiltonian operator may be obtained from the operator* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *by some ND-covering.*

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